Math 304 Midterm 1

Name: ______________________

This exam has 9 questions, for a total of 100 points.

Please answer each question in the space provided. You need to write **full solutions**. Answers without justification will not be graded. Cross out anything the grader should ignore and circle or box the final answer.

Question	Points	Score
1	18	
2	10	
3	10	
4	10	
5	10	
6	10	
7	14	
8	10	
9	8	
Total:	100	

Question 1. (18 pts)

Determine whether each of the following statements is true or false. You do NOT need to explain.

(a) In the vector space $\mathbb{R}^3$, any 4 vectors v_1, v_2, v_3, v_4 are linearly dependent.

(b) The following matrix

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is a row echelon form.

(c) Let v_1, v_2 and v_3 be three linearly independent vectors in $\mathbb{R}^4$. Then the dimension of $\mathrm{Span}(v_1, v_2, v_3)$ is 3.

(d) An $(n \times n)$ matrix is nonsingular if and only if its determinant is nonzero.

(e) We can find two $(n \times n)$ matrices A and B such that $\det(AB) \neq \det(BA)$.

(f) The linear system

$$\begin{array}{rcrcrcrcl} x & + & y & - & 3z & + & w & = & 0 \\ 2x & + & 9y & + & 7z & - & 3w & = & 0 \\ -x & + & 4y & + & 3z & - & w & = & 0 \end{array}$$

has infinitely many solutions.

Solution:

(a) True

(b) False

(c) True

(d) True

(e) False

(f) True

Question 2. (10 pts)

Solve the following linear system

$$\begin{cases} 4x_1 + 3x_2 + 2x_3 - x_4 = 4 \\ 5x_1 + 4x_2 + 3x_3 - x_4 = 4 \\ -2x_1 - 2x_2 - x_3 + 2x_4 = -3 \\ 11x_1 + 6x_2 + 4x_3 + x_4 = 11 \end{cases}$$

Solution: The augmented matrix is

$$\left[\begin{array}{rrrr|r} 4 & 3 & 2 & -1 & 4 \\ 5 & 4 & 3 & -1 & 4 \\ -2 & -2 & -1 & -2 & -3 \\ 11 & 6 & 4 & 1 & 11 \end{array}\right] \xrightarrow{-R_1+R_2} \left[\begin{array}{rrrr|r} 4 & 3 & 2 & -1 & 4 \\ 1 & 1 & 1 & 0 & 0 \\ -2 & -2 & -1 & -2 & -3 \\ 11 & 6 & 4 & 1 & 11 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2}$$

$$\left[\begin{array}{rrrr|r} 1 & 1 & 1 & 0 & 0 \\ 4 & 3 & 2 & -1 & 4 \\ -2 & -2 & -1 & -2 & -3 \\ 11 & 6 & 4 & 1 & 11 \end{array}\right] \xrightarrow{\cdots} \left[\begin{array}{rrrr|r} 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -2 & -1 & 4 \\ 0 & 0 & 1 & 2 & -3 \\ 0 & -5 & -7 & 1 & 11 \end{array}\right] \xrightarrow{-5R_2+R_4}$$

$$\left[\begin{array}{rrrr|r} 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -2 & -1 & 4 \\ 0 & 0 & 1 & 2 & -3 \\ 0 & 0 & 3 & 6 & -9 \end{array}\right] \xrightarrow{-3R_3+R_4} \left[\begin{array}{rrrr|r} 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -2 & -1 & 4 \\ 0 & 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Its row echelon form is

$$\left[\begin{array}{rrrr|r} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & -4 \\ 0 & 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

So the solutions are

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1-\alpha \\ 3\alpha + 2 \\ -3 - 2\alpha \\ \alpha \end{pmatrix}$$

Question 3. (10 pts)

Determine whether the following matrix is nonsingular. If yes, find its inverse.

$$\begin{pmatrix} 0 & 0 & 1 & -1 \\ -2 & 0 & 1 & -1 \\ 2 & 2 & 1 & 2 \\ 2 & 1 & 0 & 2 \end{pmatrix}$$

Solution: Consider

$$\left[\begin{array}{cccc|cccc} 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \\ -2 & 0 & 1 & -1 & 0 & 1 & 0 & 0 \\ 2 & 2 & 1 & 2 & 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 2 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-R_1+R_2} \left[\begin{array}{cccc|cccc} 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 2 & 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 2 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-R_2+R_3]{-R_2+R_4}$$

$$\left[\begin{array}{cccc|cccc} 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 2 & -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 2 & -1 & 1 & 0 & 1 \end{array}\right] \xrightarrow{-2R_4+R_3} \left[\begin{array}{cccc|cccc} 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & -1 & 1 & -2 \\ 0 & 1 & 0 & 2 & -1 & 1 & 0 & 1 \end{array}\right] \xrightarrow{-R_3+R_1}$$

$$\left[\begin{array}{cccc|cccc} 0 & 0 & 0 & 1 & 0 & 1 & -1 & 2 \\ -2 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & -1 & 1 & -2 \\ 0 & 1 & 0 & 2 & -1 & 1 & 0 & 1 \end{array}\right] \to \cdots$$

I skip the rest of the computation. In the end, we get

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1/2 & -1/2 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & -1 & 2 & -3 \\ 0 & 0 & 1 & 0 & 1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 & -1 & 2 \end{array}\right]$$

So the matrix is nonsingular, and its inverse is

$$\begin{pmatrix} 1/2 & -1/2 & 0 & 0 \\ -1 & -1 & 2 & -3 \\ 1 & 1 & -1 & 2 \\ 0 & 1 & -1 & 2 \end{pmatrix}$$

Question 4. (10 pts)
Compute the determinant of the following matrix

$$\begin{pmatrix} 1 & -1 & 2 & -2 \\ -1 & 2 & 1 & 6 \\ 2 & 1 & 14 & 10 \\ -2 & 6 & 10 & 33 \end{pmatrix}$$

Solution:

Apply elementary row operations and keep track of how the determinant changes.

$$\begin{vmatrix} 1 & -1 & 2 & -2 \\ -1 & 2 & 1 & 6 \\ 2 & 1 & 14 & 10 \\ -2 & 6 & 10 & 33 \end{vmatrix} = 9$$

Question 5. (10 pts)

Determine whether the following are subspaces.

(a) Is $S = \{(x, y, z) \in \mathbb{R}^3 \mid x + y \geq 0\}$ a subspace of $\mathbb{R}^3$?

Solution: Notice that $(1, 1, 0)$ is in S. Choose $\alpha = -1$, then

$$(-1) \cdot (1, 1, 0) = (-1, -1, 0)$$

is not in S. Hence, S is not closed under scalar multiplication. So S is not a subspace.

(b) Let $\mathbb{P}_2$ be the vector space of all polynomials with degree equal to or less than 2. Is

$$W = \{\text{all polynomials } p(x) = a_2x^2 + a_1x + a_0 \text{ in } \mathbb{P}_2 \text{ such that } a_1 = 0\}$$

a subspace of $\mathbb{P}_2$?

Solution: Suppose $p_1(x)$ and $p_2(x)$ are in W, that is,

$$p_1(x) = a_2x^2 + a_0$$

$$p_2(x) = b_2x^2 + b_0.$$

It follows that $p_1(x) + p_2(x) = (a_2 + b_2)x^2 + (a_0 + b_0)$. So $(p_1 + p_2)(x)$ is in W. Now suppose $p(x) = a_2x^2 + a_0 \in W$ and $\alpha \in \mathbb{R}$. Then

$$(\alpha \cdot p)(x) = (\alpha a_2)x^2 + \alpha a_0 \in W.$$

Combining these, we see that W is a subspace of $\mathbb{P}_2$.

Question 6. (10 pts)

Determine whether $x = \begin{bmatrix} 0 \\ 10 \\ 6 \\ 3 \end{bmatrix}$ lies in the span of the vectors

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 4 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 3 \\ 2 \\ 1 \end{bmatrix} \text{ and } v_3 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 3 \end{bmatrix}.$$

Solution: Consider the system

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 2 & 3 & -2 & 10 \\ 2 & 2 & 0 & 6 \\ 4 & 1 & 3 & 3 \end{array}\right]$$

Apply elementary row operations

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

It has a solution

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

So x is in the span of v_1, v_2 and v_3.

Question 7. (14 pts)

Determine whether

$$v_1 = \begin{bmatrix} 2 \\ 4 \\ 6 \\ 7 \end{bmatrix}, v_2 = \begin{bmatrix} 3 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \ v_3 = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix} \text{ and } v_4 = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 4 \end{bmatrix}$$

form a basis of $\mathbb{R}^4$.

Solution: Form the matrix by using v_1, v_2, v_3 and v_4 as column vectors

$$A = \begin{pmatrix} 2 & 3 & 0 & 2 \\ 4 & 3 & 2 & 1 \\ 6 & 0 & 0 & 3 \\ 7 & 0 & 0 & 4 \end{pmatrix}$$

Apply elementary row operations, and we get its row echelon form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

So A is nonsingular. It follows that v_1, v_2, v_3 and v_4 are linearly independent. Now observe that $\dim \mathbb{R}^4 = 4$. So $\{v_1, v_2, v_3, v_4\}$ is a basis of $\mathbb{R}^4$.

Question 8. (10 pts)

Let $\mathbb{P}_2$ be the vector space of all polynomials of degree equal to or less than 2. Determine whether the following polynomials in $\mathbb{P}_2$

$$p_1(t) = 2t + 3$$

$$p_2(t) = t^2 + t + 1$$

$$p_3(t) = (t - 1)^2$$

are linearly independent or not.

Solution:

Consider a linear combination of $(2t + 3)$, $t^2 + t + 1$ and $(t - 1)^2$:

$$c_1(2t + 3) + c_2(t^2 + t + 1) + c_3(t - 1)^2 = 0.$$

Being linearly independent is equivalent to $a_1 = a_2 = a_3 = 0$ is the only solution. Regroup the coefficients,

$$(c_2 + c_3)t^2 + (2c_1 + c_2 - 2c_3)t + (3c_1 + c_2 + c_3) = 0$$

and we have the following linear system:

$$\begin{cases} c_2 + c_3 = 0 \\ 2c_1 + c_2 - 2c_3 = 0 \\ 3c_1 + c_2 + c_3 = 0 \end{cases}$$

Solve this and we find out that

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

is the only solution. So $(2t + 3)$, $t^2 + t + 1$ and $(t - 1)^2$ are linearly independent.

Question 9. (8 pts)

(a) Suppose A is a (3×3) matrix such that $A^T = -A$. Show that $\det(A) = 0$.

Solution: We know that for any square matrix A,

$$\det(A^T) = \det(A).$$

When A is (3×3), we have

$$\det(-A) = (-1)^3 \det(A) = -\det(A).$$

Now if $A^T = -A$, then we have

$$\det(A) = \det(A^T) = \det(-A) = -\det(A).$$

So $2 \cdot \det(A) = 0$, which implies $\det(A) = 0$.

(b) Now suppose B is a (2×2) matrix such that $B^T = -B$. Then we do *not* have $\det(B) = 0$ in general. Find a (2×2) matrix B such that $B^T = -B$ and $\det(B) \neq 0$.

Solution: For example, let

$$B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Clearly, we have $\det(B) = 1 \neq 0$ and $B^T = -B$.